Higher

LOGARITHMS AND EXPONENTIALS

DR ALAN WALKER

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University of the West of Scotland

SUBJECTS STUDIED AT SCHOOL

Biology ● Chemistry ● Computing ● English
Maths ● Physics

FURTHER EDUCATION: PhD Mathematics,
BSc (Hons) Mathematics

CAREER JOURNEY SO FAR

- Teaching Fellow at the University of Strathclyde
- Research Fellow at the University of Strathclyde
- Lecturer at the University of South Wales
- Associate at Morgan Stanley
- Lecturer at the University of the West of Scotland
- Senior Lecturer at the University of the West of Scotland

FUTURE ASPIRATIONS

> Increase the size of the team of mathematicians and statisticians at the University of the West of Scotland, and provide world class mathematics-related degrees which prepare students for a world of mathematically-related employment in business, finance, engineering, science, teaching, computing...

Q&A WITH ALAN

What does your company/organisation do?

The University of the West of Scotland offers over 100-degree courses at undergraduate and postgraduate level. It also carries out research and consultancy work for industry, and is ranked second in Scotland for the number of Knowledge Transfer Partnerships with businesses. Many courses at the University of the West of Scotland have an emphasis on vocational skills and offer students the option of spending a year working in industry at home or abroad.

What types of activities do you do in your job?

Teaching: I lecture students of mathematics, physics, chemistry, engineering (mechanical, chemical, aeronautical and civil), computing, primary education, sport science, and more. Students in all of these disciplines require a working knowledge of a range of mathematical techniques, and as a mathematician, I am employed to guide and prepare them for the mathematics they might need in their future jobs.

Research: I employ mathematics in my research in liquid crystals (the materials in your flat screen displays) and in ultrasound devices. Using mathematics, I create models which can [hopefully] be used to improve such devices.

Administration: I am the Programme Leader for our BSc (Hons) Mathematics and BSc (Hons) Mathematics with Education degrees. I also sit on the Student Experience Committee, the School Plagiarism Panel, the Scottish Mathematical Council and the Scottish Mathematics Support Network. All of these roles involve making sure that our students get the best possible experience in their education, and are as prepared as possible for the challenges in the 21st Century workplace.

What does a typical day at work look like for you?

- Teaching Day: Four hours of lectures, tutorials, and maybe computer laboratory sessions with a cohort of students. Administration such as marking, creating assessments, report writing, and supporting students with queries and problems.
- Non-Teaching Day: Administration such as marking, creating assessments, report writing, and supporting students with queries and problems. Continuing with research problems, investigating ways to improve liquid crystal and ultrasound devices using mathematical modelling.

What are your favourite things about your job?

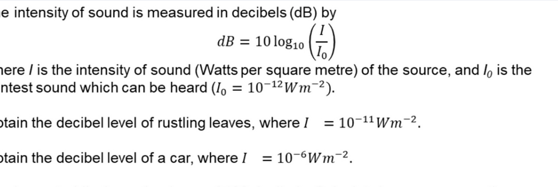
Getting to discuss mathematics with students and other scientists. Showing how mathematics is used in such a huge variety of ways in such a huge variety of sectors.

HOW ALAN USES LOGARITHMS AND EXPONENTIALS AT WORK

> Logarithms and exponentials are useful mathematical tools for describing growth or decay. Exponentiation is just a fancy name for raising to a power. An exponential function has the form $y = a$^x, and involves variable powers x of a constant base a. The base a is often assumed to be greater than one and the variable power x is known as the exponent. This function will show exponential growth for a>1 and exponential decay for a<1. What do you think it shows for a=1? If I have a maths problems which shows exponential growth (maybe like the spread of a virus in a pandemic!) then I'd be using exponentials to display the growth of the number of people with the virus.
>
> Logarithms are like the opposite of exponentials. So if I know that $y = a$^x, then I can use the logarithm function to say: x=log_ay (every mathematical function has an opposite, or inverse – logs and exponentials are the opposite of each other). I might use the logarithm function to "strip-away" the effects of an exponential function.
>
> Sometimes the base a is a special number called e, which is an irrational constant and crops up in science, and engineering. It is approximately equal to 2.728 and the function $y = e$^x and its opposite function 〖y=log〗 _ex (often written 〖y=ln〗 x) are very useful for describing exponential growth.
>
> A lot of my work is concerned with sound, and loudness is measured using decibels which makes good use of the logarithm function, as you'll see below.

ACTIVITIES

Problem 1



The intensity of sound is measured in decibels (dB) by

$$dB = 10\log_{10}\left(\frac{I}{I_0}\right)$$

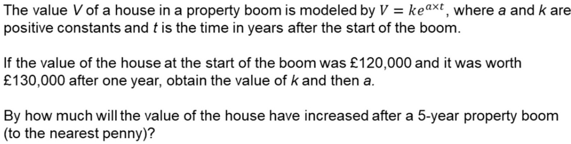
where *I* is the intensity of sound (Watts per square metre) of the source, and I_0 is the faintest sound which can be heard ($I_0 = 10^{-12}Wm^{-2}$).

Obtain the decibel level of rustling leaves, where $I = 10^{-11}Wm^{-2}$.

Obtain the decibel level of a car, where $I = 10^{-6}Wm^{-2}$.

A noisy motorbike has a loudness of 100 decibels. Calculate how many more times intense is its sound compared with a passing train at 90 decibels.

Problem 2

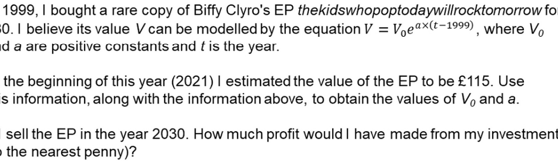


The value *V* of a house in a property boom is modeled by $V = ke^{a\times t}$, where *a* and *k* are positive constants and *t* is the time in years after the start of the boom.

If the value of the house at the start of the boom was £120,000 and it was worth £130,000 after one year, obtain the value of *k* and then *a*.

By how much will the value of the house have increased after a 5-year property boom (to the nearest penny)?

Problem 3

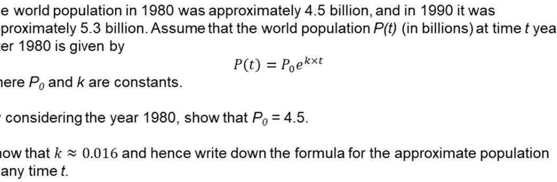


In 1999, I bought a rare copy of Biffy Clyro's EP *thekidswhopoptodaywillrocktomorrow* for £80. I believe its value *V* can be modelled by the equation $V = V_0e^{a\times(t-1999)}$, where V_0 and *a* are positive constants and *t* is the year.

At the beginning of this year (2021) I estimated the value of the EP to be £115. Use this information, along with the information above, to obtain the values of V_0 and *a*.

If I sell the EP in the year 2030. How much profit would I have made from my investment (to the nearest penny)?

Problem 4



The world population in 1980 was approximately 4.5 billion, and in 1990 it was approximately 5.3 billion. Assume that the world population *P(t)* (in billions) at time *t* years after 1980 is given by

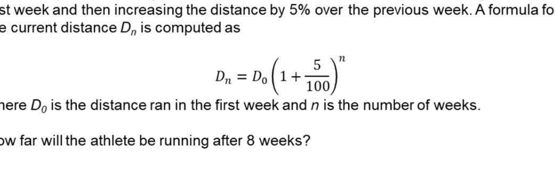
$$P(t) = P_0e^{k\times t}$$

where P_0 and *k* are constants.

By considering the year 1980, show that P_0 = 4.5.

Show that $k \approx 0.016$ and hence write down the formula for the approximate population at any time *t*.

Problem 5



An athlete plans a training schedule for the marathon which involves running 25 km in the first week and then increasing the distance by 5% over the previous week. A formula for the current distance D_n is computed as

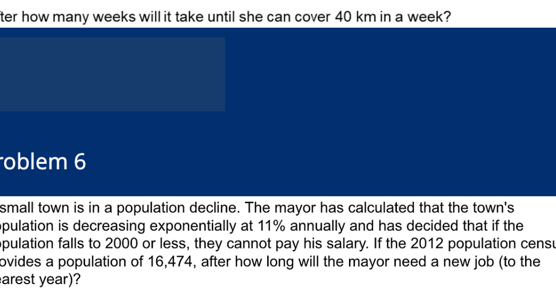
$$D_n = D_0\left(1 + \frac{5}{100}\right)^n$$

where D_0 is the distance ran in the first week and *n* is the number of weeks.

How far will the athlete be running after 8 weeks?

After how many weeks will it take until she can cover 40 km in a week?

Problem 6



A small town is in a population decline. The mayor has calculated that the town's population is decreasing exponentially at 11% annually and has decided that if the population falls to 2000 or less, they cannot pay his salary. If the 2012 population census provides a population of 16,474, after how long will the mayor need a new job (to the nearest year)?

Problem 1

Problem 1a – Solution

$$dB = 10\left(\frac{I}{I_0}\right)$$

$$dB = 10\left(\frac{10^{-11}}{10^{-12}}\right)$$

$$dB = 10\,(10)$$

$$dB = 10\times1 = 10$$

Problem 1b – Solution

$$dB = 10\left(\frac{I}{I_0}\right)$$

$$dB = 10\left(\frac{10^{-6}}{10^{-12}}\right)$$

$$dB = 10\left(10^{6}\right)$$

$$dB = 10\times6 = 60$$

Problem 2

Problem 2a – Solution

At the beginning of the boom, $t = 0$ and $V = £120,000$. Since $e^{at} = 1$, it follows that $k = V = £120,000$

After 1 year, $t = 1$ and $V = £130,000$. Substituting into the model,

$$£130,000 = £120,000e^{a\times1}$$

Solving for a,

$$13 = 12e^{a}$$

$$e^{a} = \frac{13}{12}$$

$$a = \ln ln\ \left(\frac{13}{12}\right) = 0.0800\ (to\ 3\ s.f.)$$

Problem 2b – Solution

After 5 years, substituting values from (a),

$$V = £120,000e^{0.0800t}$$

$$V = £120,000e^{0.0800\times5}$$

$$V = £120,000e^{0.0800t}$$

$$V = £120,000e^{0.0400}$$

$$V = £120,000\times1.492$$

$$V = £179,000\ (to\ 3.s.f.)$$

Problem 3

In 1999, the EP cost £80, so when t = 1999, V = £80. Hence

$$£80 = V_0e^{(a\times(1999-1999))} = V_0e^{0} = V_0.$$

Hence, V_0 =£80.
In 2021, the EP is worth £115, so when t = 2021, V =£115. Hence

$$\begin{aligned} £115 &= £80e^{(a\times(2021-1999))} \\ \frac{115}{80} &= e^{22a} \\ \Rightarrow 22a &= \ln\left(\frac{115}{80}\right) \\ \Rightarrow a &= \frac{1}{22}\ln\left(\frac{115}{80}\right) \approx 0.016. \end{aligned}$$

If I sell the EP in the year 2030. How much profit would I have made from my investment (to the nearest penny)?

$$V = £80e^{0.016\times(2030-1999)} \approx £80e^{0.511} \approx £133.41.$$

If the EP is now worth £133.41 then the profit is £53.41.

Problem 4

In 1980, $t = 0$ and $P(t) = 4.5$. Since $e^{kt} = 1$, it follows that $P_o = P = 4.5$

Problem 3b – Solution

In 1990, $t = 10$ and $P(t) = 5.3$. Substituting into the given model,

$$5.3 = 4.5e^{10k}$$

Solving for k,

$$e^{10k} = \frac{5.3}{4.5}$$

$$10k = \ln ln\ \left(\frac{5.3}{4.5}\right)$$

$$k = 0.1\ln ln\ \left(\frac{5.3}{4.5}\right) = 0.016\ to\ 2.s.f.\ as\ required$$

In general, the population at any time t after 1980 in billions is given my

$$P(t) = 4.5e^{0,016t}$$

Problem 5

Problem 5a – Solution

Given $D_0 = 25km$ and $t = 8$, substituting into the given model,

$$D_n = 25km\times\left(1 + \frac{5}{100}\right)^8$$

$$D_n = 25km\times(1.05)^8$$

$$D_n = 25km\times1.48 = 37km\ (to\ 2.s.f.)$$

Problem 5b – Solution

Given $D_0 = 25km$ and $D_n = 40km$, substituting into the given model,

$$40km = 25km\times\left(1 + \frac{5}{100}\right)^t$$

Simplifying,

$$1.6 = 1.05^t$$

Take logarithms of both sides

$$1.6 = t\times1.05$$

Solve for t

$$t = \frac{1.6}{1.05} = \frac{0.204}{0.2112} = 9.63$$

Rounding up, at the end of 10 weeks the athlete will be running 40km

Problem 6

Problem 6 – solution

We can model the town's population as

$$P\ (t) = 16474\times\ (1 - 0.11)^t = 16474\times0.89^t$$

Where t is the number of years after 2012.

Given that the town cannot pay the mayor's salary once the population falls below 2000, we can assume that this is the point where he will need a new job. To find this point, substitute $P\ (t) = 2000$:

$$2000 = 16474\times0.89^t$$

Solve for t

$$0.89^t = \frac{2000}{16474}$$

$$t0.89 = \frac{2000}{16474}$$

$$-0.0506t = -0.916$$

$$t = \frac{-0.916}{-0.0506} = 18.1\ years$$

The mayor will need a new job after 19 years, in 2031, but should probably start looking at the 18 year point!